A Comparison of Minimum Action Methods for Computing Noise-induced Transitions of the Lorenz System

Tianyu Kong Justin M. Finkel Mary Silber

The University of Chicago

Noise Induced transitions of Lorenz systems

We consider the Lorenz system with added white noise

$$\dot{X} = b(X) + \epsilon dW_t = \begin{cases} \sigma(y - x) + \epsilon dW_t^x \\ \rho x - y - xz + \epsilon dW_t^y \\ -\beta z + xy + \epsilon dW_t^z \end{cases} \quad (1)$$

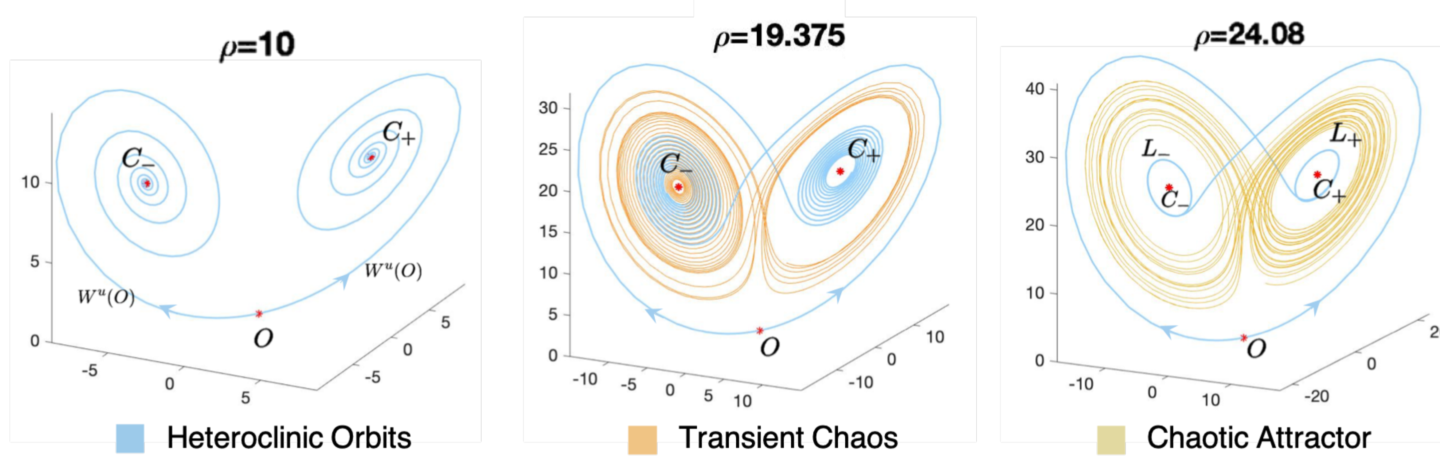

Figure 1. Phase portraits of deterministic Lorenz systems with $\sigma = 10$, $\beta = 8/3$. $C_\pm$ and O are fixed points, $L_\pm$ are limit cycles.

We want to study the transition paths of Eq (1) from stable fixed point C_- to C_+. The problem and parameter schemes are inspired by

> Xiang Zhou and Weinan E.Study of noise-induced transitions in the lorenz system using the minimum action method.Communications in Mathematical Sciences, 8(2):341–355, Jun 2010.

Numerical realizations of the transition paths

We approximate Eq (1) with Euler-Maruyama method, and plot the density of successful transitions collected.

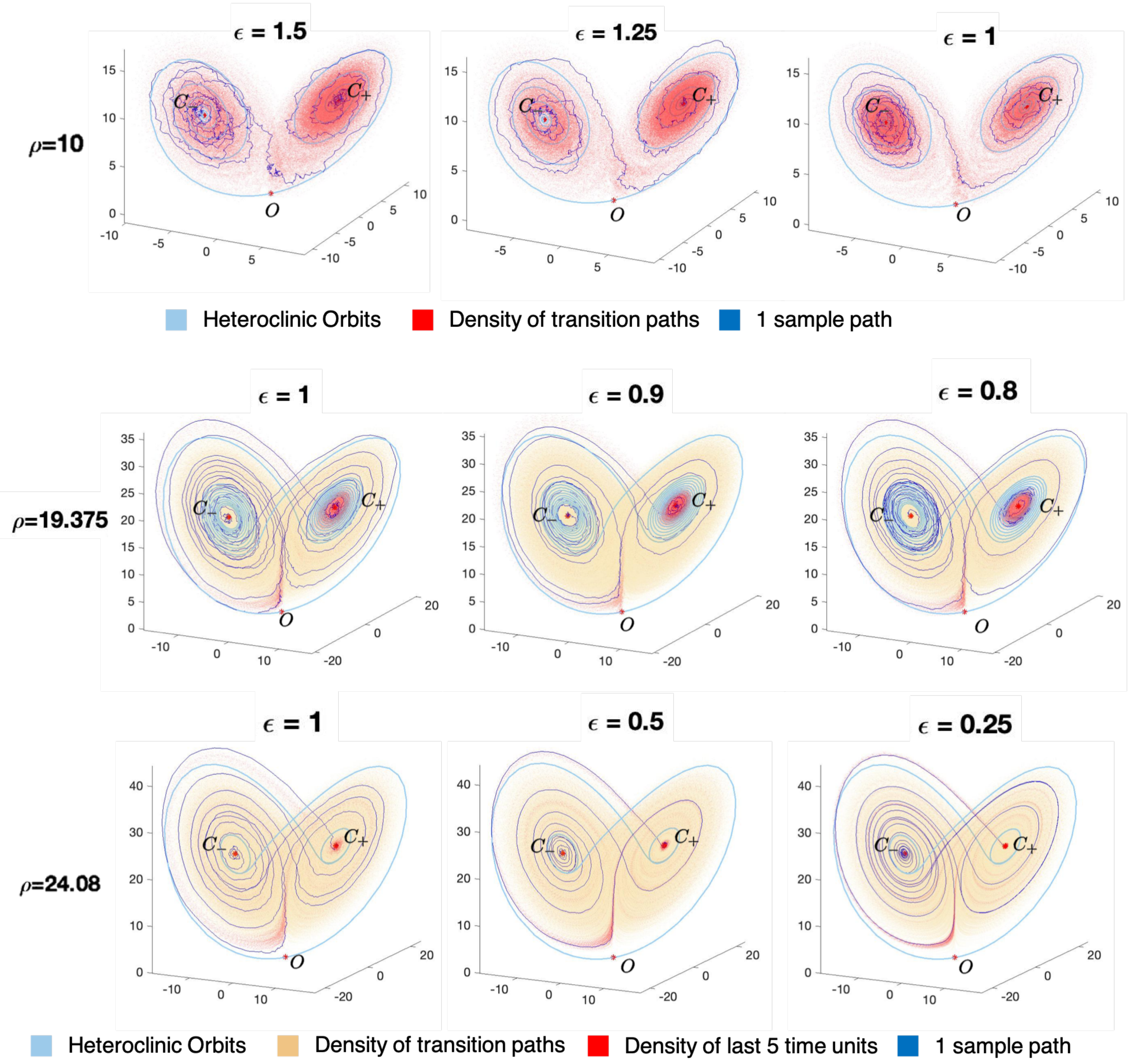

Figure 2. Collection of transition path with decreasing ϵ

Implementations of Minimum Action Method

Freidlin-Wentzell action functional

Suppose ϕ is an absolutely continuous path with $\phi(0) = C_-$, $\phi(T) = C_+$. The Minimum Action Method (MAM) states that in the zero noise limit, the Minimum Action Path (MAP) is a transition path that minimizes the Freidlin-Wentzell action functional

$$S[\phi] = \frac{1}{2}\int_0^T |\dot{\phi}(t) - b(\phi(t))|^2 dt \quad (2)$$

where b is the deterministic part of the Lorenz system in Eq (1).

We propose 2 methods of numerically computing this Minimum Action Path:

1. Calculate the gradient directly with respect to the path coordinates
2. Recast the problem in momentum coordinates, which represent the strength of noise perturbations. This greatly simplifies the objective function while complicating the endpoint constraint, which is enforced with a penalty function.

Optimization using path coordinates

Suppose $\{t_0 = 0, t_1, ..., t_N = T\}$ is a uniform discretization of time with spacing Δt. We define $x_n = \phi(t_n)$ as the discretized path. We can rewrite the Freidlin-Wentzell functional in Eq 2 as the following optimization problem:

$$\text{minimize } S(x_0, ..., x_N) = \frac{\Delta t}{2}\sum_{n=0}^{N}\left|\frac{x_{n+1} - x_n}{\Delta t} - b(x_n)\right|^2, \text{ subject to } x_0 = C_-, x_N = C_+ \quad (3)$$

The gradient of S with respect to x_n is:

$$\nabla_n S = -\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t} + [b(x_n) - b(x_{n-1})] + \frac{\Delta t}{2}\left[\frac{db(x_n)}{dx_n}\right]^T\left[\frac{x_{n+1} - x_n}{\Delta t} - b(x_n)\right], \; n = 1, \ldots, N \quad (4)$$

We then minimize the discretized action functional with limited-memory BFGS method. There are a few parameters crucial to the results of the numerical optimization, including the total time T, the discretization parameter N, and the initial guess.

Optimization using momentum coordinates

From the discretized path x_n, we define the momentum coordinates u_n, and rewrite the action functional problem in (3):

$$u_n = \frac{x_{n+1} - x_n}{\Delta t} - b(x_n), \quad S = \frac{\Delta t}{2}\sum_{n=1}^{N}|u_n|^2 \quad (5)$$

To enforce the constraint that $x_N = C_+$, we add a quadratic penalty function to the optimization problem:

$$\text{minimize } J(u_0, ..., u_N) = \frac{\Delta t}{2}\sum_{n=1}^{N}|u_n|^2 + \lambda|x_N - C_+|^2 \quad (6)$$

The gradient of J with respect to u_n can be defined recursively using the chain rule

$$\nabla_n J = u_n\Delta t + \left[\frac{d\Phi(x_N)}{dx_N}\frac{dF(x_{N-1})}{dx_{N-1}}\cdots\frac{dF(x_{n+1})}{dx_{n+1}}\Delta t\right]^T, \; n = 1, \ldots, N \quad (7)$$

$$\Phi(x_N) = \lambda|x_N - C_+|^2, \quad F(x_k) = x_k + b(x_k)\Delta t \quad (8)$$

In the optimization, we start with gradient of decreasing $u_N, u_{N-1}, ...$, and store the matrix $\frac{d\Phi(x_N)}{dx_N}\frac{dF(x_{N-1})}{dx_{N-1}}\cdots\frac{dF(x_{n+1})}{dx_{n+1}}$ to speed up calculation. We also use sequential quadratic programming on the Lagrange multiplier λ to ensure the end point constraint.

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* ISBN: 978-0-8133-4911-4 (Avalon Publishing, 2014).
2. E, W., Ren, W. & Vanden-Eijnden, E. Minimum action method for the study of rare events. *Communications on Pure and Applied Mathematics* **57**, 637–656. ISSN: 1097-0312 (2004).
3. Zhou, X. & E, W. Study of noise-induced transitions in the Lorenz system using the minimum action method. *Communications in Mathematical Sciences* **8**, 341–355. ISSN: 1539-6746, 1945-0796 (June 2010).
4. Zhou, J. X., Aliyu, M. D. S., Aurell, E. & Huang, S. Quasi-potential landscape in complex multi-stable systems. *Journal of the Royal Society Interface* **9**, 3539–3553. ISSN: 1742-5689. (2021) (Dec. 2012).

Results

Comparison of 2 optimizations

We used $T = 40$ for $\rho = 10, 19.375$ and $T = 50$ for $\rho = 24.08$. The time step is $\Delta t = 10^{-3}$. The initial guess is the line segment $\overline{C_-C_+}$ for path coordinates, and $u_n = 0$ for momentum coordinates.

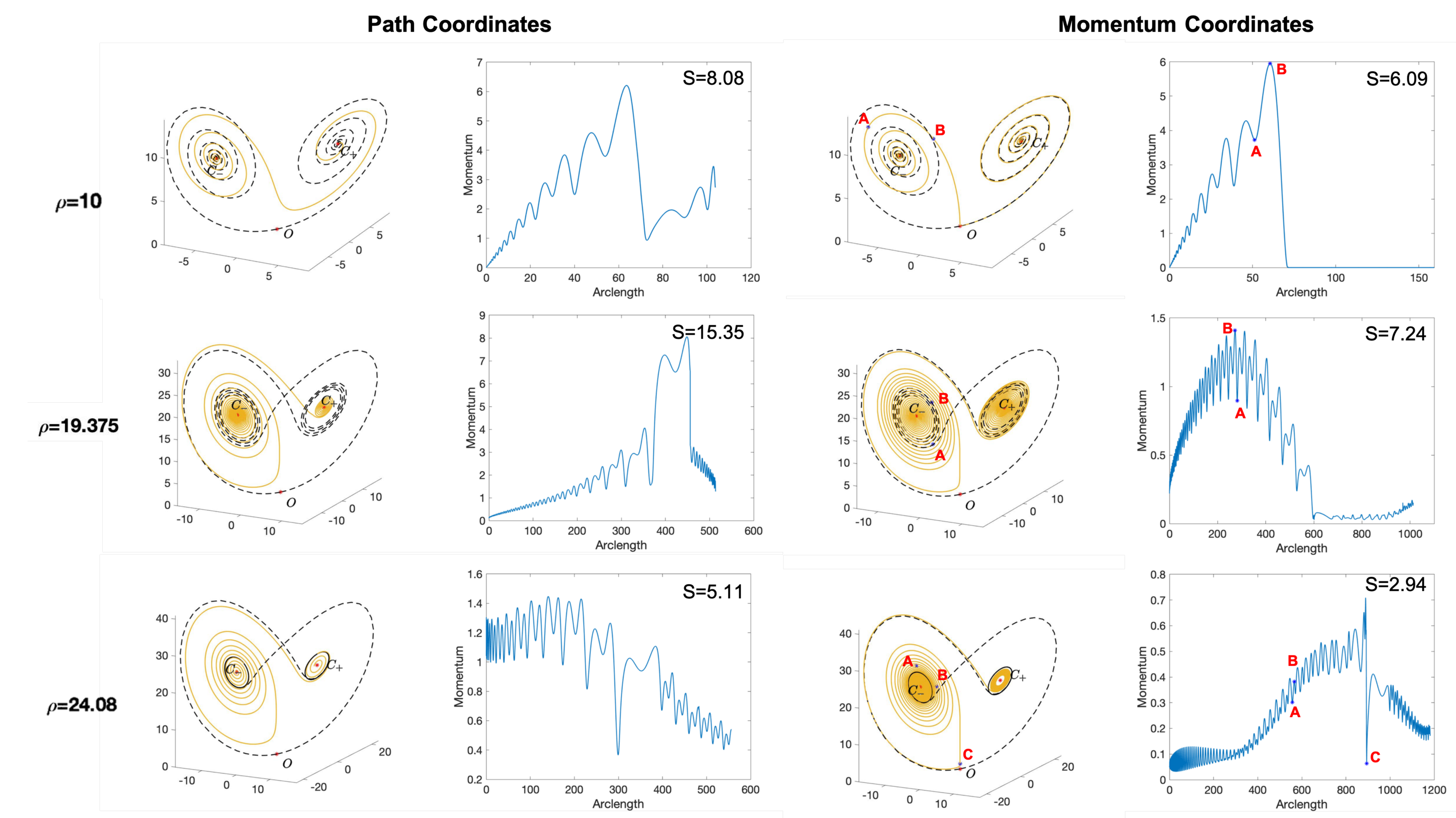

Figure 3. Optimization results of path coordinates (Left) and momentum coordinates (Right). Note the change in scale for momentum coordinates.

Quasi-potential from Minimum Action Path

The quasi-potential V measures the energy barrier between two stable fixed points. V can be defined using the Freidlin-Wentzell action functional.

$$V = \min_{\phi(0)=C_-, \phi(T)=C_+} \frac{1}{2}\int_0^T |\dot{\phi}(t) - b(\phi(t))|^2 dt \quad (9)$$

We can integrate the momentum coordinates u_n along the Minimum Action Paths to plot the energy landscape around stable fixed point C_-. We can achieve good resolution in the uphill part of the transition path for $\rho = 19.375, 24.08$. We can also map the quasi-potential along cross sections of that region.

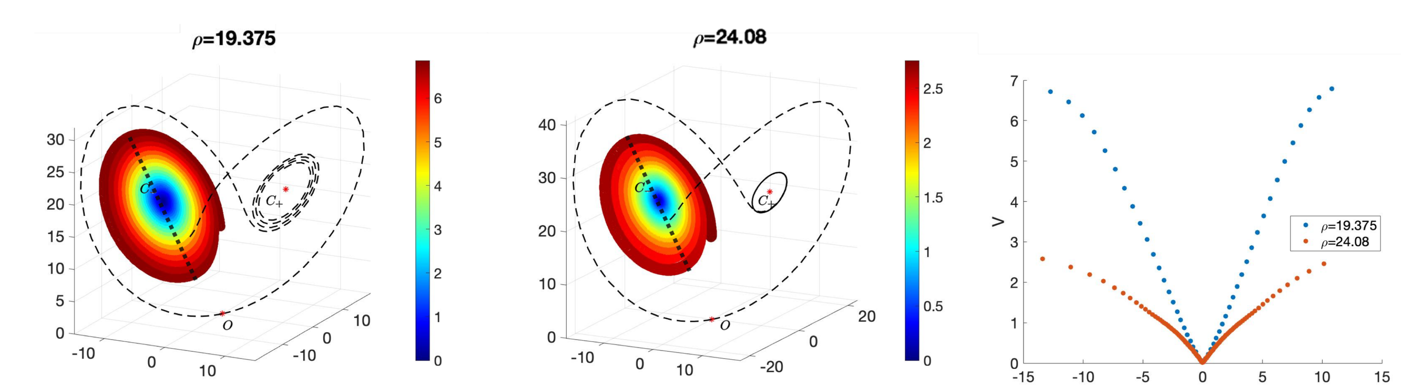

Figure 4. Quasi-potential landscape around stable fixed point C_- and the quasi-potential along one cross section. Note the change in scale for potential heat maps.

Conclusion

- We compare 2 numerical implementations to compute the Minimum Action Paths. Results from momentum coordinates have a smaller total action and only require a simple initial guess.
- We identify the MAPs passed through the origin. The overall shape of MAPs are consistent with the results from Zhou and E.
- The momentum of MAPs follows some upward trend in order to escape the potential barrier. The local minimum / maximum of perturbations in momentum coordinates correspond to the vertices of major / minor axis of elliptical shape of transition paths.
- The quasi-potential landscape of the strange attractor regime is significantly flatter than that of the transient chaos regime.